Quotations of the day:

"A mathematician is a device for turning coffee into theorems."

- Paul Erdös 1913-1996

"I have not failed. I've just found 10,000 ways that won't work."

- Thomas Edison 1847-1931

"A little inaccuracy sometimes saves a ton of explanation. "

- H. H. Munro (Saki) 1870-1916

Review:

Contrapositive

Vacuous Truth

Predicate (P(x), Q(x), etc)

Premise

Hypothesis

Inference Rules

- Definition:
  - A sequence of statements connected by $\wedge$
  - The last statement is the **conclusion**
  - The other statements are the **premises**
- Valid argument:
  - if the premises are true, the conclusion is also true
  - this must be the case for any particular set of statements substituted for the variables in the premises
    - this substitution process is call **instantiation**

Anatomy of an Inference Rule

Same as $(\mathbf{p} \vee \mathbf{q} \Rightarrow \mathbf{p}) \wedge \mathbf{q} \Rightarrow \mathbf{p}$

Major Premise: $\mathbf{p} \vee \mathbf{q} \Rightarrow \mathbf{p}$

Minor Premise: $\mathbf{q}$

Conclusion: $\therefore \mathbf{p}$

Variables: **p** and **q,** note that q is a premise *and* a variable.

Definition: An inference rule of this form with two premises followed by a conclusion is called a ***syllogism***.

Some possible inference rules:

$p \Rightarrow q$	If stocks go up I make money.
q	I have made money.
$\therefore\ p$	$\therefore$ Stocks have gone up.

Converse Error!

$p \Rightarrow q$	If I study I will pass the class.
$\sim p$	I have not studied.
$\therefore\ \sim q$	$\therefore$ I will not pass the class.

Inverse Error!

Testing An Argument

- Is an argument valid or invalid? One test is:
  - construct a truth table for the premises and the conclusion
  - find the **critical rows --** those in which all of the premises are true
  - check the value of the conclusion in these rows
    - if true for all critical rows, the argument is **VALID**
    - otherwise the argument is **INVALID**

Invalid Inference Rules: Converse Error.

$p \Rightarrow q$	If stocks go down I lose money.
q	I have lost money.
$\therefore\ p$	$\therefore$ Stocks have gone down.

Variables /--------- Premises ----------\ Conclusion

p	q	q	$p \Rightarrow q$	p
T	T	T	T	T
T	F	F	F	T
F	T	T	T (vacuous)	F
F	F	F	T (vacuous)	F

Invalid Inference Rules: Inverse Error.

$p \Rightarrow q$ — If I study I will pass the class.

$\sim p$ — I have not studied.

$\therefore \sim q$ — $\therefore$ I will not pass the class.

Variables /--------- Premises ----------\ Conclusion

p	q	~p	$p \Rightarrow q$	~q
T	T	F	T	F
T	F	F	F	T
F	T	T	T (vacuous)	F
F	F	T	T (vacuous)	T

Valid Inference Rules

- Modus Ponens
- Modus Tollens
- Disjunctive Addition
- Conjunctive Addition
- Conjunctive Simplification
- Disjunctive Syllogism
- Hypothetical Syllogism
- Contradiction Rule
- Dilemma

Modus Ponens

(the method of affirming)

Arbitrary Form	Instantiated Form
$p \Rightarrow q$	If you tickle him, he will laugh.
p	He is being tickled.
$\therefore q$	$\therefore$ He is laughing.

Valid Inference Rules: Modus Ponens

$p \Rightarrow q$ If you tickle him, he will laugh.

p He is being tickled.

$\therefore q$ $\therefore$ He is laughing.

p	***q***	***p* → *q***	***p***	***q***
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

Modus Tollens

(the method of denying)

Arbitrary Form	Instantiated Form
$p \Rightarrow q$	If you build it, they will come.
$\sim q$	They did not come.
$\therefore \sim p$	$\therefore$ You did not build it.

Valid Inference Rules: Modus Tollens

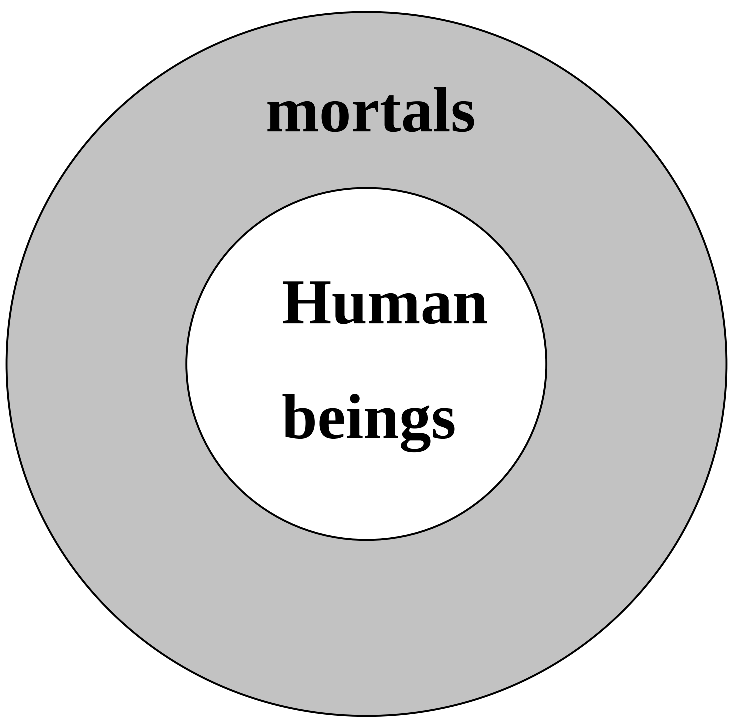
(Verifying the rule with Venn Diagrams)

Arbitrary Form	Instantiated Form	Structure
$p \Rightarrow q$	Humans are mortal.	Major Premise
$\sim q$	Zeus is not mortal.	Minor Premise
$\therefore \sim p$	$\therefore$ Zeus is not human.	Conclusion

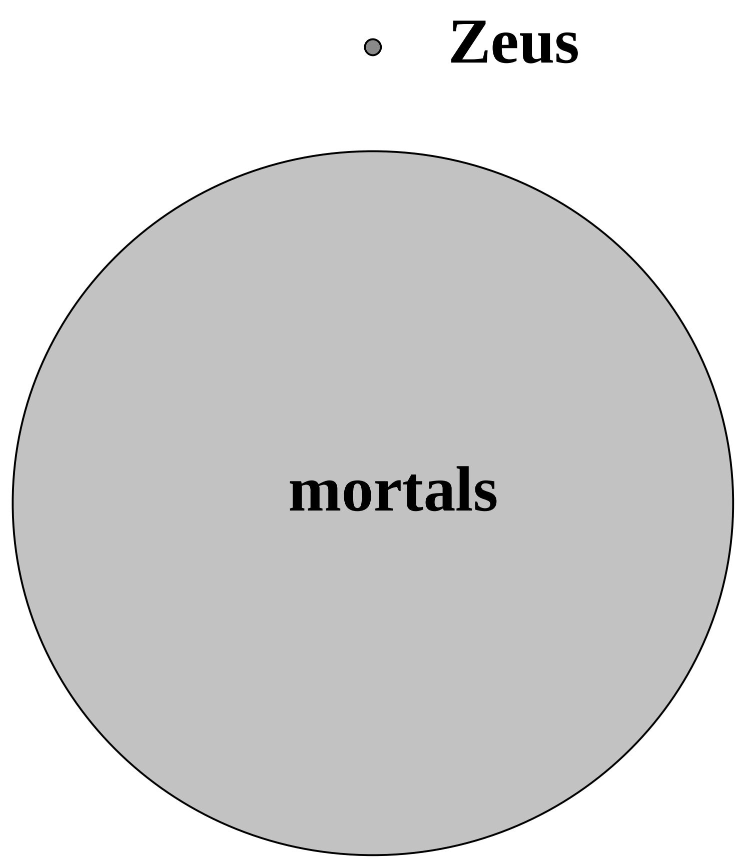
Valid Inference Rules: Modus Tollens

(Verifying the rule with Venn Diagrams)

Major Premise



Minor Premise



Valid Inference Rules: Modus Tollens

(Verifying the rule with Venn Diagrams)

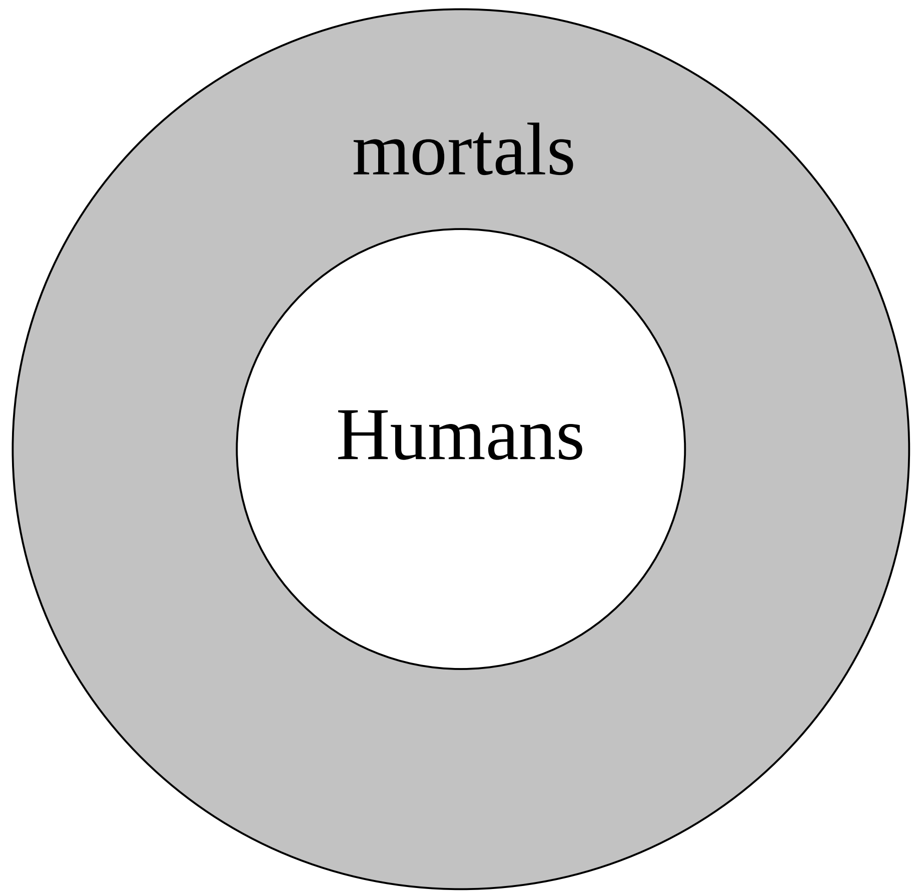
Combine the major and minor premise diagrams:

This is the conclusion.

1) Is there only one way to combine the premise diagrams?

Zeus

2) Does the combined diagram match the expected conclusion?



Practice Problems

- Use modus ponens or modus tollens to fill in the blank
  - If you do not freeze, then I will shoot.
  - You did not freeze.
  - Therefore: I shot ________________.

- Uses Modus Ponens.

Practice Problems

- Use modus ponens or modus tollens to fill in the blank
  - If they were unsure of the address, then they would have telephoned.
  - ~~They did not telephone~~.
  - Therefore, they were sure of the address.

- Uses Modus Tollens.

Practice Problems

- Use modus ponens or modus tollens to fill in the blank
  - If the moon is made of cheese, it is Wednesday.
  - The moon is made of cheese.
  - Therefore: It is Wednesday.

- Uses Modus Ponens.

Practice Problems

- Use modus ponens or modus tollens to fill in the blank
  - If sqrt(2) is rational, then sqrt(2) = a/b for some integers *a* and *b*.
  - It is not true that sqrt(2) = a/b for some integers *a* and *b*.
  - Therefore, <u>Sqrt(2) is not rational</u>_______________.

- Uses Modus Tollens.

Definition: A conjunction is another way of saying “and” or “$\wedge$.”

Definition: Disjunction is another way of saying “or” or “$\vee$.”

Disjunctive Addition

(method of generalizing)

Arbitrary Form	Instantiated Form
p	He is being tickled.
$\therefore\ p \vee q$	∴He is being tickled or he is sad
q	He is hungry.
$\therefore\ p \vee q$	∴He is hungry or he is Swiss

Conjunctive Addition

(Formalization of the definition)

Arbitrary Form	Instantiated Form
p	The dog is smelly.
q	The dog has no nose.
$\therefore\ p \wedge q$	∴The dog has no nose and smells.

Conjunctive Simplification

(method of particularization)

Arbitrary Form	Instantiated Form
$p \wedge q$	He is sad and he is eating.
$\therefore\ p$	∴He is sad.
$p \wedge q$	He is hungry and he is Swiss.
$\therefore\ q$	∴He is Swiss.

Disjunctive Syllogism

(method of “ruling-out”)

Arbitrary Form	Instantiated Form
$p \vee q$	He is sad or he is eating.
$\sim q$	He is not sad.
$\therefore\ p$	∴ He is eating.
$p \vee q$	He is hungry or he is Swiss.
$\sim p$	He is not Swiss
$\therefore\ q$	∴ He is hungry.

Valid Inference Rules: Disjunctive Syllogism

$p \vee q \vee r$;

$\sim r$;

$\therefore p \vee q$

It's red, blue, or green.

It's not green

∴ It's red or blue.

p	q	r	$p \vee q \vee r$	$\sim r$	$p \vee q$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	T	F	T
T	F	F	T	T	T
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	F	T	F

Hypothetical Syllogism

(transitivity of implication ($\Rightarrow$))

Arbitrary Form	Instantiated Form
$p \Rightarrow q$	If Henry is teething he will cry.
$q \Rightarrow r$	If Henry is crying he will not sleep.
$\therefore\ p \Rightarrow r$	$\therefore$ If Henry is teething he will not sleep.

Dilemma

(Greek, Di: “two”, lemma: “take”)

(Division into cases)

Arbitrary Form	Instantiated Form
$p \vee q$	*x* is positive or x is negative.
$p \Rightarrow r$	If x is positive then $x^2 > 0$.
$q \Rightarrow r$	If x is negative then $x^2 > 0$.
$\therefore r$	$\therefore x^2 > 0$.

Practice Problem

– Use valid inference rules to create new premises that imply the conclusion.

A: $\sim p \vee q \Rightarrow r$

B: $s \vee \sim q$

C: $\sim w$

D: $p \Rightarrow w$

E: $\sim p \wedge r \Rightarrow \sim s$

F: $\mathrm{D} \wedge \mathrm{C} \Rightarrow \sim \mathrm{p}$ (Modus Tollens)

G: $\mathrm{F} \Rightarrow \sim \mathrm{p} \vee \mathrm{q}$ (Disjunctive Addition)

H: $\mathrm{F} \wedge \mathrm{A} \Rightarrow \mathrm{r}$ (Modus Ponens)

I: $\mathrm{F} \wedge \mathrm{H} \Rightarrow \sim \mathrm{p} \wedge \mathrm{r}$ (Conjunctive Addition)

J: $\mathrm{I} \wedge \mathrm{E} \Rightarrow \sim \mathrm{s}$ (Modus Ponens)

K: $\mathrm{J} \wedge \mathrm{B} \Rightarrow \sim \mathrm{q}$ (Disjunctive Syllogism)

– Conclusion:

- Therefore, $\sim q$

Where are my glasses?

A: If my glasses are on the kitchen table then I saw them at breakfast.

B: I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

C: If I was reading the newspaper in the living room then my glasses are on the coffee table.

D: I did not see my glasses at breakfast.

E: If I was reading my book in bed then my glasses are on the the bedside table.

G: If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

Solution

If my glasses are on the kitchen table then I saw them at breakfast.

I did not see my glasses at breakfast.

∴ My glasses are not on the kitchen table.

If I was reading the paper in the kitchen, then my glasses are on the kitchen table.

My glasses are not on the kitchen table.

∴ I did not read the paper in the kitchen.

I was reading the paper in the living room or I was reading it in the kitchen.

I did not read the paper in the kitchen.

∴ I was reading the paper in the living room

If I was reading the paper in the living room then my glasses are on the coffee table.

I was reading the paper in the living room.

∴ My glasses are on the coffee table.

Solution Symbolically -- The Form

- Let *p* be “my glasses are on the kitchen table”.
- Let *q* be “I saw my glasses at breakfast”.
- Let *r* be “I was reading the newspaper in the living room”.
- Let *s* be “I was reading the newspaper in the kitchen”.
- Let *w* be “my glasses are on the coffee table”.
- Let *u* be “I was reading my book in bed”.
- Let *v* be “my glasses are on the bed table”.

A: If my glasses are on the kitchen table then I saw them at breakfast. $p \Rightarrow q$

B: I was reading the newspaper in the living room or I was reading the newspaper in the kitchen. $r \vee s$

C: If I was reading the newspaper in the living room then my glasses are on the coffee table. $r \Rightarrow w$

D: I did not see my glasses at breakfast. $\sim q$

E: If I was reading my book in bed then my glasses are on the the bedside table. $u \Rightarrow v$

G: If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table. $s \Rightarrow p$

Using the formal representation we can deduce w:

$p \Rightarrow q$

$r \lor s$

$r \Rightarrow w$

$\sim q$

$u \Rightarrow v$

$s \Rightarrow p$

$p \Rightarrow q$

$\sim q$

$\therefore \sim p$

$r \lor s$

$\sim s$

$\therefore r$

$s \Rightarrow p$

$\sim p$

$\therefore \sim s$

$r \Rightarrow w$

r

$\therefore w$

Rule Of Contradiction

- Definition:
  - If you can show that a supposition “the statement *p* is false” leads to a contradiction, then you can conclude the statement *p* is true.
- Formally:

$\sim p \Rightarrow c$ (where c is a contradiction)

$\therefore p$

Knights and Knaves

An island is inhabited by knights and knaves.

Knaves always lie.

Knights always tell the truth.

You have met two inhabitants (A and B) of the island:

> A says: B is a knight.
>
> B says: A is not the same as me.

What are A and B?

Solution

Suppose A is a Knight.

∴ What A says is true. (By definition of Knight)

∴ B is also a knight. (A said so and A tells the truth)

∴ B tells the truth. (By definition of knight)

∴ A and B are of opposite types. (B said so and B tells the truth)

… but this is a contradiction!

If A is a knight (the supposition) then it logically follows that B is also a knight. And it also follows that B is not the same type as A.

∴ The supposition is false. (By rule of contradiction)

∴ A is a knave. (By disjunctive syllogism)

∴ B is not a knight. (Since we know know A lies)

∴ B is a knave (By disjunctive syllogism)

∴ A and B are both knaves.

Valid inferences with false conclusions:

- Example:
    - If John Lennon was a rock star, then John Lennon had red hair.
    - John Lennon was a rock star.
    - Therefore, John Lennon had red hair.
  - The conclusion is false because the premise is is an incorrect statement.
  - The inference is still valid.

Invalid Argument With A True Conclusion

- Example:
    - If New York is a big city, then New York has tall buildings.
    - New York has tall buildings.
    - Therefore, New York is a big city.
  - The conclusion is a correct statement.
  - The way we got it uses an invalid argument based on the common mistake called **converse error**.

Proofs

What is a Proof?

- A proof is a formal argument for the truth of some statement.
- A proof is an algorithm for demonstrating the truth of a statement and as such is like writing a computer program.
- A proof is a sequence of premises derived from previous premises using valid inference rules.
- The last premise in the proof is the conclusion and is what was to be proven.

Anatomy of a proof:

State the proposition to be proven as formally as possible.

Proof:

Inference *(Axiom, Inference Rule, or Definition used)*

…

Inference *(Axiom, Inference Rule, or Definition used)*

Q.E.D or □

Direct Proof

$\forall x \in$ Domain D, if P(x) then Q(x)

Suppose that $x \in$ D, and that P(x) is true.

We would like to show that Q(x) can be shown from P(x).

We can use:

Definitions

Previous Proofs

Valid Inference Rules

Direct Proof 1

- If the sum of two integers is even then so is the difference of those two integers.

Formally: $\forall x, y \in \mathbf{Z}$, even$(x + y) \Rightarrow$ even$(x - y)$

Background Research:

Definition of even:

n is even if $\exists k \in \mathbf{Z} \ni n = 2k$

Definition of odd:

n is odd if $\exists k \in \mathbf{Z} \ni n = 2k + 1$

Direct Proof 1

- If the sum of two integers is even then so is the difference of those two integers.

Formally: $\forall x, y \in \mathbf{Z}, \text{even}(x + y) \Rightarrow \text{even}(x - y)$

Proof:

Let m and n be integers, such that $m + n$ is even.

$m + n = 2k$ for some integer k (By definition of even)

$m = 2k - n$ (Subtract n from both sides, algebra)

$m - n = 2k - 2n$ (Subtract n from both sides, again)

$m - n = 2(k - n)$ (Factor out the two, arithmetic)

…but $k - n$ is just some integer j (Integers closed under subtraction)

$m - n = 2j$ (Substitute j for $k - n$, algebra)

By definition of even $m - n$ is even since it has the form $2j$.

Q. E. D.

Direct Proof 2

- The sum of any two rational numbers is rational. (Closure of rational number under addition)

Formally: $\forall x, y \in \mathbf{Q}, (x + y) \in \mathbf{Q}$

Background Research:

Definition of rational number:

A rational number can be written as the quotient of two integers.

Formally: $\forall x \in \mathbf{Q}, \exists a,b \in \mathbf{Z} \ni x = a/b$

Direct Proof 2

Formally: $\forall$x, y $\in$ **R**, (x + y) $\in$ **R**

Proof:

Let *m* and *n* be rational numbers.

$$m = \frac{a}{b} \quad n = \frac{c}{d} \qquad \text{(Definition of Rational)}$$

$$m + n = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \qquad \text{(Arithmetic)}$$

Let the integer p = ad + bc **(Integers are closed over arithmetic)**

Let the integer q = bd **(q is non-zero since b and z are non-zero)**

$$m + n = \frac{p}{q}, \quad p, q \in \mathrm{Z} \wedge q \neq 0 \qquad \text{(Substitution)}$$

m + *n* is rational since it is a quotient of integers.

Q. E. D.

Proof by Counterexample

- Good for proving that a universal statement is false.

For example:

Prove that the statement "For all real numbers x and y, if $x^2 = y^2$ then $x = y$" is false.

Proof by Counterexample

Prove that: If x^2 is equal to y^2 x does not necessarily equal y.

Formally: $\sim \forall x, y \in R, (x^2 = y^2) \Rightarrow (x = y)$

Or: $\exists x, y \in R \ni (x^2 = y^2) \wedge (x \neq y)$

Proof by Counterexample:

Let *m* be the real number 3

Let *n* be the real number -3

$3^2 = (-3)^2$ i.e. $9 = 9$

$3 \neq -3$

Q. E. D.

Indirect Proofs

- What are indirect proofs

Proof by Contradiction

Approach:

1) Negate the statement to be proved.
2) Derive a contradiction using the negated statement.
3) Since the negated statement caused a paradox the negated statement cannot be true.
4) If the negated statement is false then the statement itself must be true.

Contradiction Proof 1

Prove that $\sqrt{2}$ is an irrational number

$$\text{Stated Formally}: \forall a, b \in Z, \frac{a}{b} \neq \sqrt{2}$$

$$\text{Negate the formal expression}: \exists a, b \in Z \ni \frac{a}{b} = \sqrt{2}$$

Contradiction Proof 1

Prove that $\sqrt{2}$ is irrational.

Formally: $\forall a, b \in Z, \frac{a}{b} \neq \sqrt{2}$

Proof by contradiction:

Suppose that $\sqrt{2}$ is rational, i.e. $\exists a, b \in Z \ni \frac{a}{b} = \sqrt{2}$

$\sqrt{2} = \frac{m}{n}$, where m and n are integers and have no common factors (definition of rational number)

$2 = \frac{m^2}{n^2}$ (Square both sides, algebra)

$m^2 = 2n^2$ (Multiply both sides by n^2, algebra)

m^2 is even (Since $2n^2$ is of the form 2k, where k is an integer)

Contradiction Proof 1

Prove that $\sqrt{2}$ is irrational.

Formally: $\forall a, b \in Z, \frac{a}{b} \neq \sqrt{2}$

Proof by contradiction:

$\ldots$

$m^2 = 2n^2$ (Multiply both sides by n^2, algebra)

m^2 is even (Since $2n^2$ is of the form 2k, where k is an integer)

Since m^2 is even m must also be even (Lemma)

$m = 2k$ (Definition of even)

$m^2 = (2k)^2 = 4k^2 = 2n^2$ (Substitution)

$2k^2 = n^2$ (Divide $4k^2 = 2n^2$ by 2)

$2j = n$ and is even (The square root of an even number is even)

Contradiction Proof 1

$$\text{Formally}: \forall \text{a}, \text{b} \in \text{Z}, \frac{\text{a}}{\text{b}} \neq \sqrt{2}$$

Proof by contradiction:

$\cdots$

$\sqrt{2} = \frac{m}{n}$, where m and n are integers and have no common factors (definition of rational number)

$\cdots$

$2j = n$ and is even (The square root of an even number is even)

We know now that *m* and *n* are both even. Since they are both even they share the factor 2. This contradicts our earlier premise.

Since this paradox was logically derived from the supposition that sqrt(2) is rational we know that our supposition was wrong and that sqrt(2) is irrational. (By rule of contradiction)

Q.E.D.

Fallacies

- Common mistakes we make:
  - using vague or ambiguous premises
  - assuming what is to be proved
  - jumping to conclusions
  - begging the question
- Two others that look like modus ponens and modus tollens (again)
  - converse error
    - assuming a statement is the **converse** of what is stated
  - inverse error
    - assuming a statement is the **inverse** of what is stated